

# Permutahedra, HKR isomorphism and polydifferential Gerstenhaber-Schack complex

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*To Murray Gerstenhaber and Jim Stasheff*

## 1 Introduction

This paper aims to give a short but self-contained introduction into the theory of (wheeled) props, properads, dioperads and operads, and illustrate some of its key ideas in terms of a prop(erad)ic interpretation of simplicial and permutahedra cell complexes with subsequent applications to the Hochschild-Kostant-Rosenberg type isomorphisms.

Let  $V$  be a graded vector space over a field  $\mathbb{K}$  and  $\mathcal{O}_V := \odot^\bullet V^*$  the free graded commutative algebra generated by the dual vector space  $V^* := \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$ . One can interpret  $\mathcal{O}_V$  as the algebra of polynomial functions on the space  $V$ . The classical Hochschild-Kostant-Rosenberg theorem asserts that the Hochschild cohomology of  $\mathcal{O}_V$  (with coefficients in  $\mathcal{O}_V$ ) is isomorphic to the space,  $\wedge^\bullet \mathcal{T}_V$ , of polynomial polyvector fields on  $V$  which in turn is isomorphic as a vector space to  $\wedge^\bullet V \otimes \odot^\bullet V^*$ ,

$$HC^\bullet(\mathcal{O}_V) \simeq \wedge^\bullet \mathcal{T}_V \simeq \wedge^\bullet V \otimes \odot^\bullet V^*. \quad (1)$$

The Hochschild complex,  $C^\bullet(\mathcal{O}_V) = \bigoplus_{k \geq 0} \text{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1 - k]$ , of  $\mathcal{O}_V$  has a natural subcomplex,  $C_{diff}^\bullet(\mathcal{O}_V) \subset C^\bullet(\mathcal{O}_V)$ , spanned by polydifferential operators. It was proven in [Ko] (see also [CFL]) that again

$$HC_{diff}^\bullet(\mathcal{O}_V) \simeq \wedge^\bullet \mathcal{T}_V \simeq \wedge^\bullet V \otimes \odot^\bullet V^*. \quad (2)$$

The first result (1) actually fails for a ring of smooth functions,  $\mathcal{O}_M$ , on a generic graded manifold  $M$  while the second one (2) stays always true [CFL]. Thus one must, in general, be careful in distinguishing ordinary and polydifferential Hochschild cohomology for smooth functions.

The vector space  $\mathcal{O}_V$  has a natural (co)commutative bialgebra structure so that one can also associate to  $\mathcal{O}_V$  a Gerstenhaber-Schack complex [GS1],  $C^{\bullet, \bullet}(\mathcal{O}_V) := \bigoplus_{m, n \geq 1} \text{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[2 - m - n]$ . Its cohomology was computed in [GS2, LM],

$$HC^{\bullet,\bullet}(\mathcal{O}_V) \simeq \wedge^{\bullet \geq 1} V \otimes \wedge^{\bullet \geq 1} V^*. \quad (3)$$

In this paper we introduce (more precisely, *deduce* from the permutahedra cell complex) a relatively non-obvious *polydifferential* subcomplex,  $\mathcal{C}_{diff}^{\bullet,\bullet}(\mathcal{O}_V) \subset \mathcal{C}^{\bullet,\bullet}(\mathcal{O}_V)$ , such that  $\mathcal{C}_{diff}^{\bullet,\bullet}(\mathcal{O}_V) \cap \mathcal{C}^{\bullet}(\mathcal{O}_V) = \mathcal{C}_{diff}^{\bullet}(\mathcal{O}_V)$ , and prove that this inclusion is a quasi-isomorphism,

$$HC_{diff}^{\bullet,\bullet}(\mathcal{O}_V) \simeq \wedge^{\bullet \geq 1} V \otimes \wedge^{\bullet \geq 1} V^*. \quad (4)$$

In fact, we show in this paper very simple pictorial proofs of all four results, (1)-(4), mentioned above: first we interpret Sanedlidze-Umble's [SU] permutahedra cell complex as a differential graded (dg, for short) properad  $\mathcal{P}$ , then we use tensor powers of  $\mathcal{P}$  to create a couple of other dg props,  $\mathcal{D}$  and  $\mathcal{Q}$ , whose cohomology we know immediately by their very constructions, and then, studying representations of  $\mathcal{D}$  and  $\mathcal{Q}$  in an arbitrary vector space  $V$  we *obtain* (rather than *define*) the well-known polydifferential subcomplex of the Hochschild complex for  $\mathcal{O}_V$  and, respectively, a new polydifferential subcomplex of the Gerstenhaber-Schack complex whose cohomologies are given, in view of contractibility of the permutahedra, by formulae (2) and (4). Finally, using again the language of props we deduce from (2) and (4) formulae (1) and, respectively, (3). As a corollary to (4) and (3) we show a slight sharpening of the famous Etingof-Kazhdan theorem: *for any Lie bialgebra structure on a vector space  $V$  there exists its bialgebra quantization within the class of polydifferential operators from  $\mathcal{C}_{poly}^{\bullet,\bullet}(\mathcal{O}_V)$ .*

The paper is organized as follows. In §2 we give a short but self-contained introduction into the theory of (wheeled) props, properads, dioperads and operads. In §3 we prove formulae (1)-(4) using properadic interpretation of the permutahedra cell complex. In §4 we study a dg prop,  $\mathcal{Def} \mathcal{Q}$ , whose representations in a dg space  $V$  are in one-to-one correspondence with unital  $A_\infty$ -structures on  $\mathcal{O}_V$ , and use it to give a new pictorial proof of another classical result that isomorphisms (1) and (2) extend to isomorphisms of Lie algebras, with  $\wedge^{\bullet} \mathcal{T}_M$  assumed to be equipped with Schouten brackets.

We work over a field  $\mathbb{K}$  of characteristic zero. If  $V = \bigoplus_{i \in \mathbb{Z}} V^i$  is a graded vector space, then  $V[k]$  is a graded vector space with  $V[k]^i := V^{i+k}$ . We denote  $\otimes^{\bullet} V := \bigoplus_{n \geq 0} \otimes^n V$ ,  $\otimes^{\bullet \geq 1} V := \bigoplus_{n \geq 1} \otimes^n V$ , and similarly for symmetric and skewsymmetric tensor powers,  $\odot^{\bullet} V$  and  $\wedge^{\bullet} V$ . The symbol  $[n]$  stands for an ordered set  $\{1, 2, \dots, n\}$ .

## 2 An introduction to the theory of (wheeled) props

**2.1 An associative algebra as a morphism of graphs.** Recall that an *associative algebra* structure on a vector space  $E$  is a linear map  $E \otimes E \rightarrow E$  satisfying the associativity condition,  $(a_1 a_2) a_3 = a_1 (a_2 a_3)$ , for any  $a_1, a_2, a_3 \in E$ . Let us represent a typical element,  $a_1 \otimes a_2 \otimes \dots \otimes a_n \in \otimes^n E$ , of the tensor algebra,  $\otimes^{\bullet} E$ , of  $E$  as a *decorated directed graph*,

$$G\langle a_1, \dots, a_n \rangle := \begin{array}{c} \bullet a_1 \\ | \\ \bullet a_2 \\ | \\ \vdots \\ | \\ \bullet a_n \end{array},$$

where the adjective *decorated* means that each vertex of the shown graph  $G$  is equipped with an element of  $E$  and the adjective *directed* means that the graph  $G$  is equipped with the flow running by default (unless otherwise is explicitly shown) from the bottom to the top. Let  $G\langle E \rangle$  be the vector space spanned by all such decorations,  $G\langle a_1, \dots, a_n \rangle$ , of the shown chain-like graph  $G$  modulo the relations of the type,

$$\begin{array}{c} \bullet a_1 \\ | \\ \bullet \lambda_1 a'_2 + \lambda_2 a''_2 \\ | \\ \vdots \\ | \\ \bullet a_n \end{array} = \lambda_1 \begin{array}{c} \bullet a_1 \\ | \\ \bullet a'_2 \\ | \\ \vdots \\ | \\ \bullet a_n \end{array} + \lambda_2 \begin{array}{c} \bullet a_1 \\ | \\ \bullet a''_2 \\ | \\ \vdots \\ | \\ \bullet a_n \end{array} \quad \forall \lambda_1, \lambda_2 \in \mathbb{K},$$

which identify  $G\langle E \rangle$  with  $\otimes^n E$ . Note that if  $G$  has only one internal vertex (we call such graphs (1,1)-*corollas*), then  $G\langle E \rangle = E$ . The multiplication operation in  $E$  gets encoded in this picture as a contraction of an internal edge, e.g.

$$\begin{array}{c} \bullet a_1 \\ | \\ \bullet a_2 \\ | \\ \vdots \\ | \\ \bullet a_n \end{array} \longrightarrow \begin{array}{c} \bullet a_1 a_2 \\ | \\ \vdots \\ | \\ \bullet a_n \end{array}$$

which upon repetition gives a *contraction* map  $\mu_G : G\langle E \rangle \rightarrow E$ . Moreover, the associativity conditions for the multiplication assures us that the map  $\mu_G$  is canonical, i.e. it does *not* depend on a particular sequence of contractions of the graph  $G$  into a corolla and is uniquely determined by the graph  $G$  itself.

Actually there is no need to be specific about contracting precisely *two* vertices — any connected subset of vertices will do! Denoting the set of all possible directed connected chain-like graphs with one input leg and one output leg by  $\mathfrak{G}_1^1$ , one can equivalently define an associative algebra structure on a vector space  $E$  as a collection of linear maps,  $\{\mu_G : G\langle E \rangle \rightarrow E\}_{G \in \mathfrak{G}_1^1}$ , which satisfy the condition,

$$\mu_G = \mu_{G/H} \circ \mu'_H,$$

for any subgraph  $H \subset G$ ,  $H \in \mathfrak{G}_1^1$ . Here  $\mu'_H : G\langle E \rangle \rightarrow (G/H)\langle E \rangle$  is the map which equals  $\mu_H$  on the decorated vertices lying in  $H$  and which is identity on all other vertices, while  $\mu_{G/H} : (G/H)\langle E \rangle \rightarrow E$  is the contraction map associated with the graph  $G/H$  obtained from  $G$  by contracting all vertices lying in the subgraph  $H$  into a single corolla.

**2.2 Families of directed labelled graphs.** Thus the notion of an associative algebra can be encoded into the family of graphs  $\mathfrak{G}_1^1$  with morphisms

of graphs given by contractions along (admissible) subgraphs. This interpretation of an associative algebra structure has a strong potential for generalization leading us directly to the notions of *wheeled props*, *props*, *properads*, *dioperads* and *operads* depending on the way we choose to enlarge the above rather small and primitive family of graphs  $\mathfrak{G}_1^1$ . There are several natural enlargements of  $\mathfrak{G}_1^1$ :

(i)  $\mathfrak{G}^\circ$  is, by definition, the family of *arbitrary* (not necessarily connected) directed graphs built step-by-step from the so called  $(m, n)$ -corollas,

$$\begin{array}{c} m \text{ output legs} \\ \text{---} \overbrace{\quad \quad \quad}^{\quad \quad \quad} \\ \text{---} \cdot \text{---} \\ \text{---} \underbrace{\quad \quad \quad}_n \text{ input legs} \end{array}, \quad m, n \geq 0, \quad (5)$$

by taking their disjoint unions and/or gluing some output legs of one corolla with the same number of input legs of another corolla. This is the largest possible enlargement of  $\mathfrak{G}_1^1$  in the class of *directed* graphs. We have  $\mathfrak{G}^\circ = \coprod_{m, n \geq 0} \mathfrak{G}^\circ(m, n)$ , where  $\mathfrak{G}^\circ(m, n) \subset \mathfrak{G}^\circ$  is the subset of graphs having  $m$  output legs and  $n$  input legs, e.g.

$$\begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} \in \mathfrak{G}^\circ(2, 1), \quad \begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} \in \mathfrak{G}^\circ(1, 1), \quad \begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} \in \mathfrak{G}^\circ(3, 2). \quad (6)$$

(ii)  $\mathfrak{G}_c^\circ = \coprod_{m, n \geq 0} \mathfrak{G}_c^\circ(m, n)$  is a subset of  $\mathfrak{G}^\circ$  consisting of *connected* graphs. For example, the first two graphs in (6) belong to  $\mathfrak{G}_c^\circ$  while the third one (which is the disjoint union of the first two graphs) does not.

(iii)  $\mathfrak{G}^\dagger = \coprod_{m, n \geq 0} \mathfrak{G}^\dagger(m, n)$  is a subset of  $\mathfrak{G}^\circ$  consisting of directed graphs with no *closed* directed paths of internal edges which begin and end at the same vertex, e.g. the first graph in (6) belongs to  $\mathfrak{G}^\dagger$ , while the other two do not.

(iv)  $\mathfrak{G}_c^\dagger := \mathfrak{G}^\dagger \cap \mathfrak{G}_c^\circ$ .

(v)  $\mathfrak{G}_{c,0}^\dagger$  is a subset of  $\mathfrak{G}_c^\dagger$  consisting of graphs of *genus zero* (as 1-dimensional CW complexes).

(vi)  $\mathfrak{G}^1$  is a subset of  $\mathfrak{G}_{c,0}^\dagger$  built from corollas (5) of type  $(1, n)$  only,  $n \geq 1$ . We have  $\mathfrak{G}^1 = \coprod_{n \geq 1} \mathfrak{G}^1(1, n)$  and we further abbreviate  $\mathfrak{G}^1(n) := \mathfrak{G}^1(1, n)$ ; thus  $\mathfrak{G}^1(n)$  is the subset of  $\mathfrak{G}^1$  consisting of graphs with precisely  $n$  input legs. All graphs in  $\mathfrak{G}^1$  have precisely one output leg.

Let  $\mathfrak{G}^\vee$  be any of the above mentioned families of graphs. We assume from now on that input and output legs (if any) of graphs from  $\mathfrak{G}^\vee(m, n) \subset \mathfrak{G}^\vee$  are bijectively labelled by elements of the sets  $[n]$  and  $[m]$  respectively. Hence the group  $\mathbb{S}_m \times \mathbb{S}_n$  naturally acts on the set  $\mathfrak{G}^\vee(m, n)$  by permuting the labels<sup>1</sup>.

<sup>1</sup>in the case of  $\mathfrak{G}^1$  the action of the factor  $\mathbb{S}_m$  is, of course, trivial.

**2.3 Decorations of directed labelled graphs.** Next we have to think on what to use for *decorations* of the vertices of a graph  $G \in \mathfrak{G}^\vee(m, n)$ . The presence of the family of the permutation groups  $\{\mathbb{S}_m \times \mathbb{S}_n\}_{m, n \geq 0}$  suggests the following notion: an  $\mathbb{S}$ -bimodule,  $E$ , is, by definition, a collection of graded vector spaces,  $\{E(m, n)\}_{m, n \geq 0}$ , equipped with a left action of the group  $\mathbb{S}_m$  and with a right action of  $\mathbb{S}_n$  which commute with each other. For example, for any graded vector space  $V$  the collection,  $\mathcal{E}nd(V) = \{\mathcal{E}nd(V)(m, n) := \text{Hom}(V^{\otimes n}, V^{\otimes m})\}_{m, n \geq 0}$ , is naturally an  $\mathbb{S}$ -bimodule.

Let  $E$  be an  $\mathbb{S}$ -bimodule and  $G \in \mathfrak{G}^\vee(m, n)$  an arbitrary graph. The graph  $G$  is built by definition from a number of various  $(p, q)$ -corollas constituting a set which we denote by  $\mathbf{V}(G)$  and call the set of *vertices* of  $G$ ; the set of output (resp. input) legs of a vertex  $v \in \mathbf{V}(G)$  is denoted by  $\text{Out}_v$  (resp. by  $\text{In}_v$ ). Let  $\langle [p] \rightarrow \text{Out}_v \rangle$  be the  $p!$ -dimensional vector space generated over  $\mathbb{K}$  by the set of all bijections from  $[p]$  to  $\text{Out}_v$ , i.e. by the set of all possible labeling of  $\text{Out}_v$  by integers; it is naturally a right  $\mathbb{S}_p$ -module; we define analogously a left  $\mathbb{S}_q$ -module  $\langle \text{In}_v \rightarrow [n] \rangle$  and then define a vector space,

$$E(\text{Out}_v, \text{In}_v) := \langle [p] \rightarrow \text{Out}_v \rangle \otimes_{\mathbb{S}_p} E(p, q) \otimes_{\mathbb{S}_q} \langle \text{In}_v \rightarrow [q] \rangle.$$

An element of  $E(\text{Out}_v, \text{In}_v)$  is called a *decoration of the vertex*  $v \in \mathbf{V}(G)$ . To define next a space of *decorations of a graph*  $G$  we should think of taking the tensor product of the constructed vector spaces  $E(\text{Out}_v, \text{In}_v)$  over all vertices  $v \in \mathbf{V}(G)$  but face a problem that the set  $\mathbf{V}(G)$  is unordered so that the ordinary definition of the tensor product of vector spaces does not immediately apply. The solution is to consider first *all* possible linear orderings,  $\gamma : [k] \rightarrow \mathbf{V}(G)$ ,  $k := |\mathbf{V}(G)|$ , of the set  $\mathbf{V}(G)$  and then take coinvariants,

$$\otimes_{v \in \mathbf{V}(G)} E(\text{Out}_v, \text{In}_v) := (\oplus_{\gamma} E(\text{Out}_{\gamma(1)}, \text{In}_{\gamma(1)}) \otimes \dots \otimes E(\text{Out}_{\gamma(k)}, \text{In}_{\gamma(k)}))_{\mathbb{S}_k},$$

with respect to the natural action of the group  $\mathbb{S}_k$  permuting the orderings. Now we are ready to define the vector space of *decorations* of the graph  $G$  as a quotient of the unordered tensor product,

$$G\langle E \rangle := (\otimes_{v \in \mathbf{V}(G)} E(\text{Out}_v, \text{In}_v))_{\text{Aut } G}$$

with respect to the automorphism group of the graph  $G$  which is, by definition, the subgroup of the symmetry group of the 1-dimensional  $CW$ -complex underlying the graph  $G$  which fixes the legs. An element of  $G\langle E \rangle$  is called a *graph  $G$  with internal vertices decorated by elements of  $E$* . Thus a decorated graph is essentially a pair,  $(G, [a_1 \otimes \dots \otimes a_k])$ , consisting of a graph  $G$  with  $k = |\mathbf{V}(G)|$  and an equivalence class of tensor products of elements  $a_{\bullet} \in E$ . Note that if  $E = \{E(m, n)\}$  is a *dg*  $\mathbb{S}$ -bimodule (i.e. each  $E(m, n)$  is a complex equipped with an  $\mathbb{S}_m \times \mathbb{S}_n$ -equivariant differential  $\delta$ ) then  $G\langle E \rangle$  is naturally a dg vector space with the differential

$$\delta_G(G, [a_1 \otimes \dots \otimes a_k]) := (G, [\sum_{i=1}^{k=|\mathbf{V}(G)|} (-1)^{a_1 + \dots + a_{i-1}} a_1 \otimes \dots \otimes \delta a_i \otimes \dots \otimes a_k]).$$

Note also that if  $G$  is an  $(m, n)$ -graph with one internal vertex (i.e. an  $(m, n)$ -corolla), then  $G\langle E \rangle$  is canonically isomorphic to  $E(m, n)$ .

**2.4 Props, properads, dioperads and operads** Let  $\mathfrak{G}^\vee$  be one of the families of graphs introduced in § 2.2. A subgraph  $H \subset G$  of a graph  $G \in \mathfrak{G}^\vee$  is called *admissible* if both  $H$  and  $G/H$  also belong to  $\mathfrak{G}^\vee$ , where  $G/H$  is the graph obtained from  $G$  by shrinking all vertices and all internal edges of  $H$  into a new single vertex.

**2.4.1 Definition.** A  $\mathfrak{G}^\vee$ -algebra is an  $\mathbb{S}$ -bimodule  $E = \{E(m, n)\}$  together with a collection of linear maps,  $\{\mu_G : G\langle E \rangle \rightarrow E\}_{G \in \mathfrak{G}^\vee}$ , satisfying the “associativity” condition,

$$\mu_G = \mu_{G/H} \circ \mu'_H, \quad (7)$$

for any admissible subgraph  $H \subset G$ , where  $\mu'_H : G\langle E \rangle \rightarrow (G/H)\langle E \rangle$  is the map which equals  $\mu_H$  on the decorated vertices lying in  $H$  and which is identity on all other vertices of  $G$ . If the  $\mathbb{S}$ -bimodule  $E$  underlying a  $\mathfrak{G}^\vee$ -algebra has a differential  $\delta$  satisfying, for any  $G \in \mathfrak{G}^\vee$ , the condition  $\delta \circ \mu_G = \mu_G \circ \delta_G$ , then the  $\mathfrak{G}^\vee$ -algebra is called *differential*.

**2.4.2 Remarks.** (a) For the family of graphs  $\mathfrak{G}^1$  the condition (7) is void for elements in  $E(m, n)$  with  $m \neq 1$ . Thus we may assume without loss of generality that a  $\mathfrak{G}^1$ -algebra  $E$  satisfies an extra condition that  $E(m, n) = 0$  unless  $m = 1$ . For the same reason we may assume that a  $\mathfrak{G}_1^1$ -algebra  $E$  satisfies  $E(m, n) = 0$  unless  $m = n = 1$ .

(b) As we have an obvious identity  $\mu_G = \mu_{G/G} \circ \mu'_G$ , the “associativity” condition (7) can be equivalently reformulated as follows: for any two admissible subgraphs  $H_1, H_2 \subset G$  one has

$$\mu_{G/H_1} \circ \mu'_{H_1} = \mu_{G/H_2} \circ \mu'_{H_2}, \quad (8)$$

i.e. the contraction of a decorated graph  $G$  into a decorated corolla along a family admissible subgraphs does not depend on particular choices of these subgraphs (if there are any). This is indeed a natural extension of the notion of associativity from 1 dimension to 3 dimensions, and hence we can omit double commas in the term.

**2.4.3 Definitions** (see, e.g., [MSS, Va, BM, Me2] and references cited there)

- (i) An  $\mathfrak{G}^\circ$ -algebra  $E$  is called a *wheeled prop*.
- (ii) An  $\mathfrak{G}_c^\circ$ -algebra is called a *wheeled properad*.
- (iii) An  $\mathfrak{G}^1$ -algebra is called a *prop*.
- (iv) An  $\mathfrak{G}_c^1$ -algebra is called a *properad*.
- (v) An  $\mathfrak{G}_{0,c}^1$ -algebra is called a *dioperad*.
- (vi) An  $\mathfrak{G}^1$ -algebra is called an *operad*.
- (vii) An  $\mathfrak{G}_1^1$ -algebra is called an *associative algebra*.

**2.4.4 Remarks.** (a) We have an obvious chain of inclusions of the categories of  $\mathfrak{G}^\vee$ -algebras,

$$(vii) \subset (vi) \subset (v) \subset (iv) \subset (iii) \subset (ii) \subset (i).$$

(b) Note that *every* subgraph of a graph in  $\mathfrak{G}^\circ$  is admissible. In this sense wheeled props are the most general and natural algebraic structures associated with the class of *directed* graphs. The set of independent operations in a  $\mathfrak{G}^\circ$ -algebra is generated by one vertex graphs with at least one *loop* (that is, an internal edge beginning and ending at the vertex) and by two vertex graphs without closed directed paths (i.e. the ones belonging to  $\mathfrak{G}^\uparrow$ ).

(c) By contrast to wheeled props, the set of operations in an ordinary prop, i.e. in a  $\mathfrak{G}^\uparrow$ -algebra, is generated by the set of two vertex graphs only, and, as it is not hard to check, if the associativity condition hold for three vertex graphs, then it holds for arbitrary graphs in  $\mathfrak{G}^\uparrow$ . This is *not* true for  $\mathfrak{G}^\circ$ -algebras which is a first indication that the homotopy theory for wheeled props should be substantially different from the one for ordinary props.

(d) If we forget orientations (i.e. the flow) on edges and work instead with a family of undirected graphs,  $\mathfrak{G}$ , built, by definition, from corollas with  $m \geq 1$  undirected legs via their gluings, then we get a notion of  $\mathfrak{G}$ -algebra which is closely related to the notion of *modular operad* [GK].

**2.5. First basic example: endomorphism  $\mathfrak{G}^\vee$ -algebras.** For any finite-dimensional vector space  $V$  the  $\mathbb{S}$ -bimodule  $\mathcal{E}nd_V = \{\text{Hom}(V^{\otimes n}, V^{\otimes m})\}$  is naturally a  $\mathfrak{G}^\vee$ -algebra called the *endomorphism  $\mathfrak{G}^\vee$ -algebra of  $V$* .<sup>2</sup> For any two vertex graph  $G \in \mathfrak{G}_c^\uparrow$  the associated composition  $\mu_G : G\langle \mathcal{E}nd_V \rangle \rightarrow \mathcal{E}nd_V$  is the ordinary composition of two linear maps; for a one vertex graph  $G \in \mathfrak{G}^\circ$  with say  $k$  loops the associated map  $\mu_G$  is the ordinary  $k$ -fold trace of a linear map; for a two vertex disconnected graph  $G \in \mathfrak{G}^\uparrow$  the associated map  $\mu_G$  is the ordinary tensor product of linear maps. It is easy to see that all the axioms are satisfied.

Note that for all  $\mathfrak{G}^\vee$ -algebras except  $\mathfrak{G}^\circ$  and  $\mathfrak{G}_c^\circ$  the basic algebraic operations  $\mu_G$  do *not* involve traces so that the above assumption on *finite-dimensionality* of  $V$  can be dropped for endomorphism props, properads, dioperads and operads. If  $V$  is a (finite-dimensional) *dg* vector space, then  $\mathcal{E}nd_V$  is naturally a *dg  $\mathfrak{G}^\vee$ -algebra*.

**2.6 Second basic example: a free  $\mathfrak{G}^\vee$ -algebra.** For an  $\mathbb{S}$ -bimodule,  $E = \{E(m, n)\}$ , one can construct another  $\mathbb{S}$ -bimodule,  $\mathcal{F}^\vee\langle E \rangle = \{\mathcal{F}^\vee\langle E \rangle(m, n)\}$  with

$$\mathcal{F}^\vee\langle E \rangle(m, n) := \bigoplus_{G \in \mathfrak{G}^\vee(m, n)} G\langle E \rangle.$$

This  $\mathbb{S}$ -bimodule  $\mathcal{F}^\vee\langle E \rangle$  has a natural  $\mathfrak{G}^\vee$ -algebra structure with the contraction maps  $\mu_G$  being tautological. The  $\mathfrak{G}^\vee$ -algebra  $\mathcal{F}^\vee\langle E \rangle$  is called the *free  $\mathfrak{G}^\vee$ -algebra* (i.e., respectively, *the free wheeled prop*, *the free prop*, *the free dioperad* etc) *generated by the  $\mathbb{S}$ -bimodule  $E$* .

**2.7 Morphisms of  $\mathfrak{G}^\vee$ -algebras.** A morphism of  $\mathfrak{G}^\vee$ -algebras,  $\rho : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ , is a morphism of the underlying  $\mathbb{S}$ -bimodules such that, for any graph

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<sup>2</sup>For  $\mathfrak{G}^1$ -algebras, that is for operads, it is enough to restrict oneself to the case  $m = 1$  only, i.e. set, by default,  $\mathcal{E}nd_V := \{\text{Hom}(V^{\otimes n}, V)\}_{n \geq 1}$  (cf. §2.4.2(a)).

$G \in \mathfrak{G}^\vee$ , one has  $\rho \circ \mu_G = \mu_G \circ (\rho^{\otimes G})$ , where  $\rho^{\otimes G}$  means a map,  $G\langle \mathcal{P}_1 \rangle \rightarrow G\langle \mathcal{P}_2 \rangle$ , which changes decorations of each vertex in  $G$  in accordance with  $\rho$ . It is often assumed by default that a morphism  $\rho$  is homogeneous which (almost always) implies that  $\rho$  has degree 0. Unless otherwise is explicitly stated we do *not* assume in this paper that morphisms of  $\mathfrak{G}^\vee$ -algebras are homogeneous so that they can have nontrivial parts in degrees other than zero. A morphism of  $\mathfrak{G}^\vee$ -algebras,  $\mathcal{P} \rightarrow \mathcal{E}nd\langle V \rangle$ , is called a *representation* of the  $\mathfrak{G}^\vee$ -algebra  $\mathcal{P}$  in a graded vector space  $V$ . If  $\mathcal{P}_1$  is a free  $\mathfrak{G}^\vee$ -algebra,  $\mathcal{F}^\vee\langle E \rangle$ , generated by some  $\mathbb{S}$ -bimodule  $E$ , then the set of morphisms  $\mathfrak{G}^\vee$ -algebras,  $\{\rho : \mathcal{P}_1 \rightarrow \mathcal{P}_2\}$ , is in one-to-one correspondence with the set of morphisms of  $\mathbb{S}$ -bimodules,  $\{\rho|_E : E \rightarrow \mathcal{P}_2\}$ , i.e. a  $\mathfrak{G}^\vee$ -morphism is uniquely determined by its values on the generators. In particular, the set of morphisms,  $\mathcal{F}^\vee\langle E \rangle \rightarrow \mathcal{P}_2$ , has a graded vector space structure for any  $\mathcal{P}_2$ .

A *free resolution* of a dg  $\mathfrak{G}^\vee$ -algebra  $\mathcal{P}$  is, by definition, a dg free  $\mathfrak{G}^\vee$ -algebra,  $(\mathcal{F}^\vee\langle E \rangle, \delta)$ , generated by some  $\mathbb{S}$ -bimodule  $E$  together with a degree zero morphism of dg  $\mathfrak{G}^\vee$ -algebras,  $\pi : (\mathcal{F}^\vee\langle E \rangle, \delta) \rightarrow \mathcal{P}$ , which induces a cohomology isomorphism. If the differential  $\delta$  in  $\mathcal{F}^\vee\langle E \rangle$  is decomposable with respect to compositions  $\mu_G$ , then  $\pi : (\mathcal{F}^\vee\langle E \rangle, \delta) \rightarrow \mathcal{P}$  is called a *minimal model* of  $\mathcal{P}$ . In this case the free algebra  $\mathcal{F}^\vee\langle E \rangle$  is often denoted by  $\mathcal{P}_\infty$ .

**2.8 Props and properads.** We shall work in this paper only with  $\mathfrak{G}^\dagger$ - and  $\mathfrak{G}_c^\dagger$ -algebras, i.e. with props and properads. For later use we mention several useful constructions with these graph-algebras.

(i) There is a functor,  $\Psi$ , which associates canonically to an arbitrary dg properad,  $\mathcal{P}$ , an associated dg prop  $\Psi(\mathcal{P})$  [Va]. As we are working over a field of characteristic 0, this functor is, by Künneth theorem, exact, i.e.  $\Psi(H(\mathcal{P})) = H(\Psi(\mathcal{P}))$ . For example, if  $\mathcal{P}$  is a dg free properad  $(\mathcal{F}_c^\dagger\langle E \rangle, \delta)$ , then  $\Psi(\mathcal{P})$  is precisely  $\mathcal{F}^\dagger\langle E \rangle$  with the same differential (as given on the generators).

(ii) The above mentioned functor,  $\mathcal{F}^\dagger : (E, \delta) \rightarrow (\mathcal{F}^\dagger\langle E \rangle, \delta)$ , in the category of dg  $\mathbb{S}$ -bimodules is also exact,  $H(\mathcal{F}^\dagger\langle E \rangle) = \mathcal{F}^\dagger\langle H(E) \rangle$ . Moreover, if we set in this situation  $\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle)$  for the vector space of all possible representations,  $\{\rho : \mathcal{F}^\dagger\langle E \rangle \rightarrow \mathcal{E}nd_V\} \simeq \text{Hom}(E, \mathcal{E}nd_V)$ , and define a differential  $\delta$  in  $\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle)$  by the formula  $\delta\rho := \rho \circ \delta$ , then the resulting functor,  $(E, \delta) \rightarrow (\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle), \delta)$ , in the category of complexes is exact, i.e.

$$H(\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle)) = \text{Rep}_V(H(\mathcal{F}^\dagger\langle E \rangle)) = \text{Rep}_V(\mathcal{F}^\dagger\langle H(E) \rangle). \quad (9)$$

Indeed, as we are working over a field of characteristic zero, we can always choose an equivariant chain homotopy between complexes  $(E, \delta)$  and  $(H(E), 0)$ . This chain homotopy induces a chain homotopy between complexes  $(\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle), \delta)$  and  $(\text{Rep}_V(\mathcal{F}^\dagger\langle H(E) \rangle), 0)$  proving formula (9).

(iii) There is also a natural *parity change* functor,  $\Pi$ , which associates with a dg prop(erad)  $\mathcal{P}$  a dg prop(erad)  $\Pi\mathcal{P}$  with the following property: every representation of  $\Pi\mathcal{P}$  in a graded vector space  $V$  is equivalent to a representation of  $\mathcal{P}$  in  $V[1]$ . This functor is also exact. If, for example,  $\mathcal{P}$  is a



dg free prop  $\mathcal{F}^\dagger\langle E \rangle$  generated by an  $\mathbb{S}$ -bimodule  $E = \{E(m, n)\}$ , then, as it is not hard to check,  $\Pi\mathcal{P} = \mathcal{F}^\dagger\langle \tilde{E} \rangle$ , where  $\tilde{E} := \{sgn_m \otimes E(m, n) \otimes sgn_n[m - n]\}$  and  $sgn_m$  stands for the one-dimensional sign representation of  $\mathbb{S}_m$ .


### 3 Simplicial and permutahedra cell complexes as dg properads

**3.1 Simplices as a dg properad.** A geometric  $(n-1)$ -simplex,  $\Delta_{n-1}$ , is, by definition, a subset in  $\mathbb{R}^n = \{x^1, \dots, x^n\}$ ,  $n \geq 1$ , satisfying the equation  $\sum_{i=1}^n x^i = 1$ ,  $x^i \geq 0$  for all  $i$ . To define its cell complex one has to choose an orientation on  $\Delta_{n-1}$  which is the same as to choose an orientation on the hyperplane  $\sum_{i=1}^n x^i = 1$ . We induce it from the standard orientation on  $\mathbb{R}^{n+1}$  by requiring that the manifold with boundary defined by the equation  $\sum_{i=1}^n x^i \leq 1$  is naturally oriented. Let  $(C_\bullet(\Delta_{n-1}) = \oplus_{k=0}^{n-1} C_{-k}(\Delta_{n-1}), \delta)$  be the standard (non-positively graded) cell complex of  $\Delta_{n-1}$ . By definition,  $C_{-k}(\Delta_{n-1})$  is a  $\binom{n}{k}$ -dimensional vector space spanned by  $k$ -dimensional cells,

$$\Delta_{n-1}^I := \{(x^1, \dots, x^n) \in \Delta_{n-1} | x^i = 0, i \in I\},$$

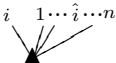
parameterized by all possible subsets  $I$  of  $[n]$  of cardinality  $n - k - 1$  and equipped with the natural orientations (which we describe explicitly below).

Note that the action,  $(x^1, \dots, x^n) \rightarrow (x^{\sigma(1)}, \dots, x^{\sigma(n)})$ , of the permutation group  $\mathbb{S}_n$  on  $\mathbb{R}^n$  leaves  $\Delta_{n-1}$  invariant as a subset but *not* as an oriented manifold with boundary. As an  $\mathbb{S}_n$ -module, one can obviously identify  $C_{1-n}(\Delta_{n-1})$  with  $sgn_n[n-1]$ , and hence one can represent pictorially the oriented cell  $\Delta_{n-1}^\emptyset$

as a labelled  $(0, n)$ -corolla, , with the symmetry condition

$$\begin{array}{c} 1 \ 2 \ \dots \ n \\ \diagup \ \diagdown \end{array} = (-1)^\sigma \begin{array}{c} \sigma(1) \sigma(2) \ \dots \ \sigma(n) \\ \diagup \ \diagdown \end{array}, \quad \forall \sigma \in \mathbb{S}_n. \quad (10)$$


The boundary of  $\Delta_{n-1}^\emptyset$  is a union of  $n$  cells,  $\Delta_{n-1}^i$ ,  $i = 1, \dots, n$ , of dimension  $n-2$ . The permutation group  $\mathbb{S}_n$  permutes, in general, these cells and changes their natural orientations while keeping their linear span  $C_{2-n}(\Delta_{n-1})$  invariant. It is obvious that the subgroup  $G_i := \{\sigma \in \mathbb{S}_n \mid \sigma(i) = i\} \simeq \mathbb{S}_{n-1}$  of  $\mathbb{S}_n$  is a symmetry group of the cell  $\Delta_{n-1}^i$  as an *unoriented* manifold with boundary. If we take the orientation into account, then the vector subspace of  $C_{2-n}(\Delta_{n-1})$  spanned by  $\Delta_{n-1}^i$  can be identified as an  $\mathbb{S}_{n-1}$ -module with  $sgn_{n-1}$ , and hence the  $n$ -dimensional space  $C_{2-n}(\Delta_{n-1})$  itself can be identified as an  $\mathbb{S}_n$ -module with  $\mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_{n-1}} sgn_{n-1}[n-2]$ . Its basis elements,  $\Delta_{n-1}^i$ , can be pictorially

represented as  $(0, n)$ -corollas  with the legs in the right bunch being skew symmetric with respect to the change of labellings by an element  $\sigma \in G_i$  (cf. (10)). The boundary operator  $\delta : C_{1-n}(\Delta_{n-1}) \rightarrow C_{2-n}(\Delta_{n-1})$  is

equivariant with respect to the  $\mathbb{S}_n$ -action and is given on the generators by the formula,

$$\delta \begin{array}{c} 1 \ 2 \ \dots \ n \\ \diagup \ \diagdown \\ \bullet \end{array} = \sum_{i=1}^n (-1)^{i+1} \begin{array}{c} i \ 1 \ \dots \ \hat{i} \ \dots \ n \\ \diagup \ \diagdown \\ \bullet \end{array}, \quad \hat{i} \text{ omitted},$$

More generally, the symmetry group of, say, a cell  $\Delta_{1-n}^I \in C_{1-k-n}(\Delta_{n-1})$  with  $I = \{i_1 < i_2 < \dots < i_{n-k}\}$  is  $\mathbb{S}_k \times \mathbb{S}_{n-k} \subset \mathbb{S}_n$  with  $\mathbb{S}_k \times \text{Id}$  leaving the orientation of  $\Delta_{n-1}^I$  invariant and  $\text{Id} \times \mathbb{S}_{n-k}$  changing the orientation via the sign representation. Thus we can identify the oriented cell  $\Delta_{n-1}^I$ , as an element

of the  $\mathbb{S}_n$ -module  $C_{1-k-n}(\Delta_{n-1})$ , with a  $(0, n)$ -corolla, , which has ‘symmetric’ output legs in the left bunch and ‘skewsymmetric’ output legs in the right one. Here  $\{j_1 < \dots < j_{n-k}\} := [n] \setminus I$ . The  $\mathbb{S}_n$ -module,  $C_{1-k-n}(\Delta_{n-1})$  is then canonically isomorphic to  $E_k(n) := \mathbb{K}[\mathbb{S}_n]_{\mathbb{S}_k \times \mathbb{S}_{n-k}}(\mathbb{1}_k \otimes \text{sgn}_{n-k})$ , where  $\mathbb{1}_k$  stands for the trivial 1-dimensional representation of  $\mathbb{S}_k$ . The boundary operator  $\delta : C_{1-k-n}(\Delta_{n-1}) \rightarrow C_{2-k-n}(\Delta_{n-1})$  is equivariant with respect to the  $\mathbb{S}_n$ -action and is given on the generators by the formula,

$$\delta \begin{array}{c} 1 \ \dots \ k \ k+1 \ \dots \ n \\ \diagup \ \diagdown \\ \bullet \end{array} = \sum_{i=k+1}^n (-1)^{i+1} \begin{array}{c} 1 \ \dots \ k \ i \ k+1 \ \dots \ \hat{i} \ \dots \ n \\ \diagup \ \diagdown \\ \bullet \end{array} \quad (11)$$

Thus we proved the following

**3.1.1 Proposition.** (i) *The standard simplicial cell complex is canonically isomorphic to a dg free properad,  $\mathcal{S} := \mathcal{F}_c^\dagger\langle E \rangle$ , generated by an  $\mathbb{S}$ -bimodule,  $E = \{E(m, n)\}$ ,*

$$E(m, n) = \begin{cases} \bigoplus_{k=0}^{m-1} E_k(m)[m-k-1] = \text{span}\langle \begin{array}{c} i_1 \ \dots \ i_k \ j_1 \ \dots \ j_{m-k} \\ \diagup \ \diagdown \\ \bullet \end{array} \rangle_{0 \leq k \leq m-1} & \text{for } n = 0, \\ 0 & \text{for } n \geq 1, \end{cases}$$

and equipped with the differential given on the generators by (11).

(ii) *The cohomology of  $(\mathcal{S}, \delta)$  is concentrated in degree zero and equals the free properad generated by the following degree zero graphs with ‘symmetric’ legs,*

$$\begin{array}{c} 1 \ 2 \ \dots \ m \\ \diagup \ \diagdown \\ \bullet \end{array} := \sum_{\sigma \in \mathbb{S}_m} \begin{array}{c} \sigma(1) \ \dots \ \sigma(m-1) \ \sigma(m) \\ \diagup \ \diagdown \\ \bullet \end{array}, \quad m \geq 1. \quad (12)$$

Claim 3.1.1(ii) follows from the contractibility of simplices, and, for each  $m$ , graph (12) represents the sum of all vertices of  $\Delta_{m-1}$ .




**3.1.2 From simplicia to Koszul complex.** The vector space,  $\text{Rep}_V(\mathcal{S})$ , of representations,  $\rho : \mathcal{S} \rightarrow \text{End}_V$ , of the simplicial properad in a vector space

$V$  can be identified with  $\sum_{k=0}^{m-1} \odot^k V \otimes \wedge^{m-k} V[m-k-1]$ . We can naturally make the latter into complex by setting  $d\rho := \rho \circ \partial$  (cf. §2.8ii). It is easy to see that we get in this way, for each  $m \geq 1$ ,

$$\wedge^m V \xrightarrow{d} V \otimes \wedge^{m-1} V \xrightarrow{d} \odot^2 V \otimes \wedge^{m-2} V \xrightarrow{d} \dots \xrightarrow{d} \odot^{m-1} V \otimes V,$$

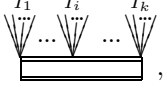
the classical Koszul complex. Hence Proposition 3.1.1(ii) and isomorphism (9) imply the well-known result that its cohomology is concentrated in degree zero and equals  $\odot^n V$ . Thus the Koszul complex is nothing but a representation of the simplicial cell complex in a particular vector space  $V$ .

**3.2 Permutahedra as a dg properad.** An  $(n-1)$ -dimensional *permutahedron*,  $P_{n-1}$ , is, by definition, the convex hull of  $n!$  points  $(\sigma(1), \sigma(2), \dots, \sigma(n))$ ,  $\forall \sigma \in \mathbb{S}_n$ , in  $\mathbb{R}^n = \{x^1, \dots, x^n\}$ . To define its cell complex one has to choose an orientation on  $P_{n-1}$  which is the same as to choose an orientation on the hyperplane  $\sum_{i=1}^n x^i = n(n+1)/2$  to which  $P_{n-1}$  belongs. We induce it from the standard orientation on  $\mathbb{R}^n$  by requiring that the manifold with boundary defined by the equation  $\sum_{i=1}^n x^i \leq n(n+1)/2$  is naturally oriented. Let  $(C_\bullet(P_{n-1}) = \oplus_{k=0}^{n-1} C_{-k}(P_{n-1}), \delta)$  stand for the associated (non-positively graded) complex of oriented cells of  $P_{n-1}$ . Its  $(n-k-1)$ -dimensional cells,  $P_{n-1}^{I_1, \dots, I_p}$ , are indexed by all possible partitions,  $[n] = I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$ , of the ordered set  $[n]$  into  $k$  disjoint ordered nonempty subsets (see [SU]). The natural action,  $(x^1, \dots, x^n) \rightarrow (x^{\sigma(1)}, \dots, x^{\sigma(n)})$ , of the permutation group  $\mathbb{S}_n$  on  $\mathbb{R}^n$  leaves  $P_{n-1}$  invariant, and hence makes the cell complex  $C_\bullet(P_{n-1})$  into an  $\mathbb{S}_n$ -module. We obviously have, for example,  $C_{1-n}(P_{n-1}) = \text{sgn}_n$ , so that we can identify the top cell  $P_{n-1}^{[n]}$  as an element of the  $\mathbb{S}_n$ -module with the  $(n, 0)$ -

corolla  with skewsymmetric output legs,   $= (-1)^\sigma$  ,  $\forall \sigma \in \mathbb{S}_n$ . More generally, a simple analysis (similar to the simplicial case in §3.1) of how the action of  $\mathbb{S}_n$  on  $\mathbb{R}^n$  permutes the cells and changes their orientation implies that  $C_{1-k-n}(P_{n-1})$  is canonically isomorphic as an  $\mathbb{S}_n$ -module to

$$W_k(n) := \bigoplus_{[n]=I_1 \sqcup I_2 \sqcup \dots \sqcup I_k} \mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_{I_1} \times \dots \times \mathbb{S}_{I_k}} (\text{sgn}_{I_1} \otimes \dots \otimes \text{sgn}_{I_k}) [k+n-1]$$

and that the cells  $P_{n-1}^{I_1, \dots, I_p}$  can be identified as elements of the  $\mathbb{S}_n$ -module with

the  $(n, 0)$ -corollas, , where legs in each  $I_i$ -bunch are skewsymmetric and the labels from  $I_i$  are assumed to be distributed over them in the increasing order from the left to the right. The boundary operator  $\delta : C_{1-k-n}(P_{n-1}) \rightarrow C_{2-k-n}(P_{n-1})$  is given on generators by (cf. [SU])

$$\delta \left( \text{graph with } I_1, \dots, I_i, \dots, I_k \text{ inputs} \right) = \sum_{i=1}^k \sum_{\substack{I_i = I'_i \sqcup I''_i \\ |I'_i|, |I''_i| \geq 1}} (-1)^{\varepsilon + \sigma_{I'_i \sqcup I''_i}} \left( \text{graph with } I_1, \dots, I'_i, I''_i, \dots, I_k \text{ inputs} \right) \quad (13)$$

where  $\varepsilon := i + 1 + I_1 + \dots + I_{i-1} + I'_i$  and  $(-1)^{\sigma_{I'_i \sqcup I''_i}}$  is the sign of the permutation  $[I_i] \rightarrow I'_i \sqcup I''_i$ . Thus we proved the following

**3.2.1 Proposition.** (i) *The Sanedblidze-Umble permutahedra cell complex is canonically isomorphic to a dg free properad,  $\mathcal{P}_\bullet := \mathcal{F}_c^\uparrow(W)$ , generated by an  $\mathbb{S}$ -bimodule,  $W = \{W(m, n)\}$ ,*

$$W(m, n) := \begin{cases} \bigoplus_{k=1}^m W_k(m) = \text{span} \langle \text{graph with } I_1, \dots, I_i, \dots, I_k \text{ inputs} \rangle & \text{for } n = 0, \\ 0 & \text{for } n \geq 1, \end{cases}$$

and equipped with the differential given on the generators by (13).

(ii) *The cohomology of  $(\mathcal{P}_\bullet, \partial)$  is concentrated in degree zero and equals a free properad generated by the following degree zero graphs,*

$$\text{graph with inputs } 1, 2, \dots, m \text{ and a black square} := \sum_{\sigma \in \mathbb{S}_m} \left( \text{graph with inputs } I_1 = \sigma(1), I_i = \sigma(i), I_m = \sigma(m) \right), \quad m \geq 1.$$

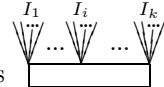
Claim 3.2.1(ii) follows from the contractibility of permutahedra, and, for each  $n$ , the above graph represents the sum of all vertices of  $P_{n-1}$ .

**3.2.2 From permutahedra to a cobar construction.** Baranovsky made in [Ba] a remarkable observation that the permutahedra cell complex can be used to compute the cohomology of the cobar construction,  $\Omega(\wedge^\bullet V)$ , where  $\wedge^\bullet V \simeq \odot^\bullet(V[1])$  is interpreted as a graded commutative coalgebra generated by a vector space  $V[1]$ . In our approach this result follows immediately from the following two observations: (i) the graded space  $\Omega(\wedge^\bullet V)$  can be identified with  $\mathbb{K} \oplus \text{Rep}_V(\mathcal{P}_\bullet)$ , where  $\text{Rep}_V(\mathcal{P}_\bullet)$  is the space of all possible representations,  $\rho : \mathcal{P}_\bullet \rightarrow \text{End}_V$ ; the differential in  $\Omega(\wedge^\bullet V)$  induced from  $\mathcal{P}_\bullet$  by the formula  $d\rho := \rho \circ \delta$  is precisely the differential of the cobar construction. Then Proposition 3.2.2(ii) and isomorphism (9) imply that the cohomology,  $H(\Omega(\wedge^\bullet V))$ , of the cobar construction equals  $\mathbb{K} \oplus \odot^{\bullet \geq 1} V = \odot^\bullet V$ .

**3.2.3 Permutahedra cochain complex.** Exactly in the same way as in §3.2 one can construct a dg properad,  $(\mathcal{P}^\bullet, \delta)$ , out of the permutahedra *cochain* complex,  $C^\bullet(P_{n-1}) := \text{Hom}_{\mathbb{K}}(C_\bullet(P_{n-1}), \mathbb{K})$ , with the differential  $\delta$  dual to the one given in (13). Very remarkably, Chapoton has shown in [Ch] that one can make the  $\mathbb{S}$ -module  $\{C^\bullet(P_{n-1})\}$  into a dg *operad*, and that this operad is a quadratic one. We do not use this interesting fact in our paper and continue interpreting instead permutahedra as a dg *properad* from which we shall build

below more complicated dg props with nice geometric and/or algebraic meaning. Let us apply first the parity chain functor,  $\Pi$ , to the dg properad  $\mathcal{P}^\bullet$  (see §2.8iii). The result,  $\mathcal{P} := \Pi\mathcal{P}^\bullet$ , is a dg free properad,  $\mathcal{F}_c^\dagger\langle\tilde{W}\rangle$ , generated by an  $\mathbb{S}$ -bimodule,  $\tilde{W} = \{\tilde{W}(m, n)\}$  with  $\tilde{W}(m, n) = 0$  for  $n \neq 0$  and with  $\tilde{W}(m, 0)$  equal to

$$Y(m) := \bigoplus_{k=1}^m \bigoplus_{\substack{[m]=I_1 \sqcup I_2 \sqcup \dots \sqcup I_k \\ |I_\bullet| \geq 1}} \mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_{I_1} \times \dots \times \mathbb{S}_{I_k}} (\mathbb{1}_{I_1} \otimes \dots \otimes \mathbb{1}_{I_k})[k]. \quad (14)$$

If we represent the generators of  $\mathcal{P}$  by corollas  with *symmetric* legs in each  $I_i$ -bunch, then the induced differential in  $\mathcal{P}$  is given by,

$$\delta \left( \text{corolla with } k \text{ bunches of legs } I_1, \dots, I_k \right) = \sum_{i=1}^k (-1)^{i+1} \left( \text{corolla with } k \text{ bunches of legs } I_1, \dots, I_i \sqcup I_{i+1}, \dots, I_k \right) \quad (15)$$

**3.2.4 Theorem.** *The cohomology of the dg properad  $\mathcal{P}$  is a free properad generated by the degree  $-m$  corollas with skewsymmetric legs,*

$$\text{corolla with } m \text{ legs } 1, 2, \dots, m := \sum_{\sigma \in \mathbb{S}_m} (-1)^\sigma \left( \text{corolla with } m \text{ legs } I_1=\sigma(1), I_i=\sigma(i), I_m=\sigma(m) \right), \quad m \geq 1.$$

**Proof.** By exactness of the functor  $\Pi$ , the statement follows from Proposition 3.2.1(ii).  $\square$

**3.2.5 Corollary.** *The cohomology of the bar construction,  $B(\odot^\bullet V)$ , of the graded commutative algebra generated by a vector space  $V$ , is equal to  $\wedge^\bullet V$ .*

**Proof.** By definition (see, e.g., [Ba]),  $B(\odot^\bullet V)$  is a free tensor coalgebra,  $\odot^\bullet(\odot^{\geq 1} V)[-1])$  with the differential  $d$  induced from the ordinary multiplication in  $\odot^{\geq 1} V$ . On the other hand, it is easy to see that as a vector space  $B(\odot^\bullet V)$  can be identified with  $\mathbb{K} \oplus \text{Rep}_V(\mathcal{P})$ , where  $\text{Rep}_V(\mathcal{P})$  is the space of all possible representations,  $\rho : \mathcal{P} \rightarrow \text{End}_V$ , of the parity shifted properad of permutahedra cochains. Moreover, the bar differential  $d$  is given precisely by  $d\rho := \rho \circ \delta$ . Thus isomorphism (9) and Theorem 3.2.4 imply the required result.  $\square$

**3.3 From permutahedra to polydifferential Hochschild complex.** Let us consider a dg  $\mathbb{S}$ -module,  $D = \{D(m, n)\}_{m \geq 1, n \geq 0}$ , with  $D(m, n) := Y(m) \otimes \mathbb{1}_n[-1]$  and with the differential,  $\delta : D(m, n) \rightarrow D(m, n)$  being equal to (15) on the tensor factor  $Y(m)$  and identity on the factor  $\mathbb{1}_n[-1]$ . Let  $(\mathcal{D} := \mathcal{F}^\dagger\langle D \rangle, \delta)$  be the associated dg free prop. Its generators can be identified with  $(m, n)$ -corollas

$$\text{corolla with } m \text{ legs } I_1, \dots, I_k \text{ and } n \text{ legs } 1, 2, 3, \dots, n \quad (16)$$

of degree  $1 - k$ , one such a corolla for every partition  $[m] = I_1 \sqcup \dots \sqcup I_k$ . The differential  $\delta$  in  $\mathcal{D}$  is then given by

$$\delta \left( \begin{array}{c} I_1 \quad I_i \quad I_{i+1} \quad I_k \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \hline 1 \quad 2 \quad 3 \quad \dots \quad n \end{array} \right) = \sum_{i=1}^{k-1} (-1)^{i+1} \begin{array}{c} I_1 \quad I_i \sqcup I_{i+1} \quad I_k \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \hline 1 \quad 2 \quad 3 \quad \dots \quad n \end{array} \quad (17)$$

**3.3.1 Proposition.** *The cohomology of the dg prop  $\mathcal{D}$  is a free prop,  $\mathcal{F}^\dagger \langle X \rangle$ , generated by an  $\mathbb{S}$ -bimodule  $X = \{X(m, n)\}_{m \geq 1, n \geq 0}$  with  $X(m, n) := \text{sgn}_m \otimes \mathbb{1}_n[m - 1]$ .*

**Proof.** By §2.8(ii), the functor  $\mathcal{F}^\dagger$  is exact so that we have  $H(\mathcal{D}) = \mathcal{F}^\dagger \langle H(\mathcal{D}) \rangle$ . By Künneth theorem,  $H(D(m, n)) = H(Y(m)) \otimes \mathbb{1}_n[-1]$ . Finally, by Theorem 3.2.4,  $H(Y(m)) = H(\mathcal{P})(m, 0) = \text{sgn}_m[m]$ .  $\square$

The space of all representations,  $\rho : \mathcal{D} \rightarrow \mathcal{E}nd_V$ , of the prop  $\mathcal{D}$  in a (finite-dimensional) vector space  $V$  can be obviously identified with

$$\text{Rep}_V(\mathcal{D}) := \bigoplus_{k \geq 1} \text{Hom}(\odot^\bullet V, (\odot^{\bullet \geq 1} V)^{\otimes k})[1 - k] = \bigoplus_{k \geq 1} \text{Hom}(\bar{\mathcal{O}}_V^{\otimes k}, \mathcal{O}_V)[1 - k],$$

where  $\mathcal{O}_V := \odot^\bullet V^*$  is the graded commutative ring of polynomial functions on the space  $V$ , and  $\bar{\mathcal{O}}_V := \odot^{\bullet \geq 1} V^*$  is its subring consisting of functions vanishing at 0. Differential (17) in the prop  $\mathcal{D}$  induces a differential,  $\delta$ , in the space  $\text{Rep}(\mathcal{D})$  by the formula,  $\delta \rho := \rho \circ \delta$ ,  $\forall \rho \in \text{Rep}(\mathcal{D})$ .

**3.3.2 Proposition.** *The complex  $(\text{Rep}(\mathcal{D})_V, \delta)$  is canonically isomorphic to the polydifferential subcomplex,  $(C_{\text{diff}}^{\bullet \geq 1}(\mathcal{O}_V), d_H)$  of the standard Hochschild complex,  $(C^\bullet(\mathcal{O}_V), d_H)$ , of the algebra  $\mathcal{O}_V$ .*

**Proof.** We shall construct a degree 0 isomorphism of vector spaces,  $i : \text{Rep}_V(\mathcal{D}) \rightarrow C_{\text{diff}}^{\bullet \geq 1}(\mathcal{O}_V)$ , such that  $i \circ \delta = d_H \circ i$ , where  $d_H$  stands for the Hochschild differential. Let  $\{e_\alpha\}$  be a basis of  $V$ , and  $\{x^\alpha\}$  the associated dual basis of  $V^*$ . Any  $\rho \in \text{Rep}_V(\mathcal{D})$  is uniquely determined by its values,

$$\rho \left( \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \hline 1 \quad 2 \quad \dots \quad n \end{array} \right) =: \sum_{\substack{I_1, \dots, I_k, J \\ |I_\bullet| \geq 1, |J| \geq 0}} \Gamma_J^{I_1, \dots, I_k} x^J \otimes e_{I_1} \otimes \dots \otimes e_{I_k} \in \text{Hom}(\bar{\mathcal{O}}_V^{\otimes k}, \mathcal{O}_V),$$

for some  $\Gamma_J^{I_1, \dots, I_k} \in \mathbb{K}$ . Here the summation runs over multi-indices,  $I = \alpha_1 \alpha_2 \dots \alpha_{|I|}$ ,  $x^I := x^{\alpha_1} \odot \dots \odot x^{\alpha_{|I|}}$ , and  $e_I := e_{\alpha_1} \odot \dots \odot e_{\alpha_{|I|}}$ . Then the required map  $i$  is given explicitly by

$$i(\rho) := \sum_{I_1, \dots, I_k, J} \frac{1}{|J|! |I_1|! \dots |I_k|!} \Gamma_J^{I_1, \dots, I_k} x^J \frac{\partial^{|I_1|}}{\partial x^{I_1}} \otimes \dots \otimes \frac{\partial^{|I_k|}}{\partial x^{I_k}}$$

where  $\partial^{|I|} / \partial x^I := \partial^{|I|} / \partial x^{\alpha_1} \dots \partial x^{\alpha_{|I|}}$ . Now it is an easy calculation to check (using the definition of the Hochschild differential) that  $i \circ \delta = d_H \circ i$ .  $\square$

**3.3.3 Corollary.**  $H(C_{diff}^{\bullet \geq 1}(\mathcal{O}_V)) = \wedge^{\bullet \geq 1} V \otimes \odot^{\bullet} V^*.$

**Proof.** By Proposition 3.3.2, the cohomology  $H(C_{diff}^{\bullet \geq 1}(\mathcal{O}_V))$  of the polydifferential Hochschild is equal to the cohomology of the complex  $(\text{Rep}_V(\mathcal{D}), \delta)$ . The latter is equal, by isomorphism (9), to  $\text{Rep}_V(H(\mathcal{D}))$  which in turn is equal, by Proposition 3.3.1, to  $\wedge^{\bullet \geq 1} V \otimes \odot^{\bullet} V^*.$   $\square$

The complex  $C_{diff}^{\bullet}(\mathcal{O}_V)$  is a direct sum,  $\mathcal{O}_V \oplus C_{diff}^{\bullet \geq 1}(\mathcal{O}_V)$ , where  $\mathcal{O}_V$  is a trivial subcomplex. Thus Corollary 3.3.3 implies isomorphism (2).

**3.4 Hochschild complex.** Is there a dg prop whose representation complex is the general (rather than polydifferential) Hochschild complex for polynomial functions? Consider a dg free prop,  $\hat{\mathcal{D}}$ , which is generated by the same  $\mathbb{S}$ -bimodule  $D$  as the prop  $\mathcal{D}$  above, but equipped with a different differential,  $\hat{\delta}$ , given on the generators by (17) and two extra terms,

$$(\hat{\delta} - \delta) \left( \begin{array}{c} I_1 \quad I_i \quad I_k \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ 1 \quad 2 \quad 3 \quad \dots \quad n \end{array} \right) = - \sum_{\substack{[n] = J_1 \sqcup J_2 \\ |J_1| = |I_1|}} \left( \begin{array}{c} I_1 \\ \vdots \\ J_1 \end{array} \right) \left( \begin{array}{c} I_2 \quad I_i \quad I_k \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ J_2 \end{array} \right) - \sum_{\substack{[n] = J_1 \sqcup J_2 \\ |J_2| = |I_k|}} (-1)^k \left( \begin{array}{c} I_1 \quad I_i \quad I_{k-1} \quad I_k \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ J_1 \quad J_2 \end{array} \right)$$

**3.4.1 Proposition.**  $H(\hat{\mathcal{D}}) = H(\mathcal{D}) = \mathcal{F}^\dagger \langle X \rangle.$

**Proof.** Define a *weight* of a generating corolla (16) of the prop  $\hat{\mathcal{D}}$  to be  $\sum_{i=1}^k |I_i|$ , and the *weight*,  $w(G)$ , of a decorated graph  $G$  from  $\hat{\mathcal{D}}$  to be the sum of weights of all constituent corollas of  $G$ . Then  $F_p := \{\text{span} \langle G \rangle \mid w(G) \leq p\}_{p \geq 0}$  is a bounded below exhaustive filtration of the complex  $(\hat{\mathcal{D}}, \hat{\delta})$ . By the classical convergence theorem, the associated spectral sequence  $\{E_r, d_r\}_{r \geq 0}$  converges to  $H(\hat{\mathcal{D}})$ . Its 0th term,  $(E_0, d_0)$ , is precisely the complex  $(\mathcal{D}, \delta)$ . Thus  $E_1$ , is, by Proposition 3.3.1, the free prop  $\mathcal{F}^\dagger \langle X \rangle$  so that  $d_1$  vanishes and the spectral sequence degenerates at the first term completing the proof.  $\square$

The complex  $(\text{Rep}_V(\hat{\mathcal{D}}), \hat{\delta})$  associated, by §2.8(ii), to the dg prop  $(\hat{\mathcal{D}}, \hat{\delta})$  is easily seen to be precisely the standard Hochschild complex,  $(C^{\bullet \geq 1}(\mathcal{O}_V) = \oplus_{k \geq 1} \text{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V), d_H)$  of the non-unital algebra  $\mathcal{O}_V$  with coefficients in the unital algebra  $\mathcal{O}_V$ . Hence Proposition 3.4.1 and isomorphism (9) immediately imply that  $HC^{\bullet \geq 1}(\mathcal{O}_V) = \wedge^{\bullet \geq 1} V \otimes \odot^{\bullet} V^*$  which in turn implies, with the help of the theory of simplicial modules (see, e.g., Proposition 1.6.5 in [Lo]), the Hochschild-Kostant-Rosenberg isomorphism (1). Using language of dg props, we deduced it, therefore, from the permutahedra cell complex.

**3.5 From permutahedra to polydifferential Gerstenhaber-Schack complex.** Let us consider a dg  $\mathbb{S}$ -module,  $Q = \{Q(m, n)\}_{m \geq 1, n \geq 1}$ , with  $Q(m, n) := Y(m) \otimes Y(n)^*[-2]$  and the differential,  $d = \delta \otimes \text{Id} + \text{Id} \otimes \delta^*$ , where  $\delta$  is given by (15). Let  $(\mathcal{Q} := \mathcal{F}^\dagger \langle Q \rangle, d)$  be the associated dg free prop.

Its generators can be identified with corollas  $\begin{array}{c} I_1 \quad I_i \quad I_m \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ J_1 \quad J_j \quad J_n \end{array}$  of degree  $2 - m - n$

with symmetric legs in each input and output bunch. The differential  $d$  is given on the generators by

$$d \left( \begin{array}{c} I_1 \quad I_i \quad I_{i+1} \quad I_m \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ J_1 \quad J_j \quad J_{j+1} \quad J_n \end{array} \right) = \sum_{i=1}^{m-1} (-1)^i \left( \begin{array}{c} I_1 \quad I_i \sqcup I_{i+1} \quad I_m \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ J_1 \quad J_j \quad J_{j+1} \quad J_n \end{array} \right) + \sum_{j=1}^{n-1} (-1)^j \left( \begin{array}{c} I_1 \quad I_i \quad I_{i+1} \quad I_m \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ J_1 \quad J_j \sqcup J_{j+1} \quad J_n \end{array} \right).$$

**3.5.1 Proposition.**  $H(\mathcal{Q})$  is a free prop generated by an  $\mathbb{S}$ -bimodule  $\{sgn_m \otimes sgn_n[m+n-2]\}_{m,n \geq 1}$ .

**Proof.** Use Theorem 3.2.4 and the Künneth theorem.  $\square$

The complex of representations,  $(\text{Rep}(\mathcal{Q}), d)$ , is isomorphic as a graded vector space to  $C^{\bullet, \bullet}(\bar{\mathcal{O}}_V) := \bigoplus_{m,n \geq 1} \text{Hom}(\bar{\mathcal{O}}_V^{\otimes m}, \bar{\mathcal{O}}_V^{\otimes n})[m+n-2]$ . The latter has a well-known *Gerstenhaber-Schack differential* [GS1],

$$d_{GS} : \text{Hom}(\bar{\mathcal{O}}_V^{\otimes n}, \bar{\mathcal{O}}_V^{\otimes m}) \xrightarrow{d_{GS}^1 \oplus d_{GS}^2} \text{Hom}(\bar{\mathcal{O}}_V^{\otimes n+1}, \bar{\mathcal{O}}_V^{\otimes m}) \oplus \text{Hom}(\bar{\mathcal{O}}_V^{\otimes n}, \bar{\mathcal{O}}_V^{\otimes m+1}),$$

with  $d_{GS}^1$  given on an arbitrary  $\Phi \in \text{Hom}(V^{\otimes n}, V^{\otimes m})$  by

$$(d_{GS}^1 \Phi)(f_0, \dots, f_n) := -\Delta^{m-1}(f_0) \cdot \Phi(f_1, \dots, f_n) + \sum_{i=0}^{n-1} (-1)^i f(f_0, \dots, f_i f_{i+1}, \dots, f_n) \\ + (-1)^n \Phi(f_1, f_2, \dots, f_{n-1}) \cdot \Delta^{m-1}(f_n), \quad \forall f_0, f_1, \dots, f_n \in \bar{\mathcal{O}}_V,$$

where the multiplication in  $\bar{\mathcal{O}}_V$  is denoted by juxtaposition, the induced multiplication in the algebra  $\bar{\mathcal{O}}_V^{\otimes m}$  by  $\cdot$ , the comultiplication in  $\bar{\mathcal{O}}_V$  by  $\Delta$ , and

$$\Delta^{m-1} : (\Delta \otimes \text{Id}^{\otimes m-2}) \circ (\Delta \otimes \text{Id}^{\otimes m-3}) \circ \dots \circ \Delta : \bar{\mathcal{O}}_V \rightarrow \bar{\mathcal{O}}_V^{\otimes m},$$

for  $m \geq 2$  while  $\Delta^0 := \text{Id}$ . The expression for  $d_{GS}^2$  is an obvious dual analogue of the one for  $d_{GS}^1$ .

It is evident, however, that  $(\text{Rep}(\mathcal{Q}), d) \neq (C^{\bullet, \bullet}(\bar{\mathcal{O}}_V), d_{GS})$ . What is then the meaning of the *naturally* constructed complex  $(\text{Rep}(\mathcal{Q}), d)$ ?

**3.5.2 Definition-proposition.** Let  $C_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V)$  be a subspace of  $C^{\bullet, \bullet}(\bar{\mathcal{O}}_V)$  spanned by polydifferential operators of the form,

$$\Phi : \begin{array}{ccc} \bar{\mathcal{O}}_V^{\otimes m} & \longrightarrow & \bar{\mathcal{O}}_V^{\otimes n} \\ f_1 \otimes \dots \otimes f_m & \longrightarrow & \Gamma(f_1, \dots, f_m), \end{array}$$

with  $\Phi(f_1, \dots, f_m) = x^{J_1} \otimes \dots \otimes x^{J_n} \cdot \Delta^{n-1} \left( \frac{\partial^{|I_1|} f_1}{\partial x^{I_1}} \right) \cdot \dots \cdot \Delta^{n-1} \left( \frac{\partial^{|I_n|} f_m}{\partial x^{I_n}} \right)$  for some families of nonempty multi-indices  $I_\bullet$  and  $J_\bullet$ . Then  $C_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V)$  is a subcomplex of the Gerstenhaber-Schack complex  $(C^{\bullet, \bullet}(\bar{\mathcal{O}}_V), d_{GS})$ .

**Proof.** Proving that  $C_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V)$  is a subcomplex of  $(C^{\bullet, \bullet}(\bar{\mathcal{O}}_V), d_{GS}^1)$  is very similar to the Hochschild complex case. So we omit these details and concentrate instead on showing that  $C_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V)$  is a subcomplex of  $(C^{\bullet, \bullet}(\bar{\mathcal{O}}_V), d_{GS}^2)$ .



If, for arbitrary  $f \in \bar{\mathcal{O}}_V$ , we use Sweedler's notation,  $\Delta f = \sum f' \otimes f''$ , for the coproduct in  $\bar{\mathcal{O}}_V$ , then, for an operator  $\Phi$  as above, one has,

$$\begin{aligned} d_{GS}^2 \Phi(f_1, \dots, f_m) &= - \sum f'_1 \cdots f'_m \otimes \Phi(f''_1, \dots, f''_m) - \sum_{i=1}^n (-1)^i \Delta_i \Phi(f_1, \dots, f_m) \\ &\quad + (-1)^n \sum \Phi(f'_1, \dots, f'_m) \otimes f''_1 \cdots f''_m \end{aligned}$$

where  $\Delta_i$  means  $\Delta$  applied to the  $i$ -tensor factor in the space of values,  $\bar{\mathcal{O}}_V^{\otimes n}$ , of  $\Phi$ . Taking into account the particular structure of  $\Phi$ , one can see that  $d_{GS}^2 \Phi$  is a linear combination of polydifferential operators if and only if an equality holds,

$$\Delta^2 \frac{\partial^{|I|} f}{\partial x^I} = \sum f' \otimes \Delta \frac{\partial^{|I|} f''}{\partial x^I},$$

for arbitrary  $f \in \bar{\mathcal{O}}_V$  and arbitrary non-empty multi-index  $I$ . As product and coproduct in  $\bar{\mathcal{O}}_V$  are consistent, it is enough to check this equality under the assumption that  $\dim V = 1$  in which case it is straightforward.  $\square$

**3.5.3 Proposition.** (i) *The complexes  $(\text{Rep}(\mathcal{Q}), d)$  and  $(C_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V), d_{GS})$  are canonically isomorphic.* (ii)  $HC_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V) = \wedge^{\bullet \geq 1} V \otimes \odot^{\bullet \geq 1} V^*$ .

**Proof.** (i) Any representation  $\rho \in \text{Rep}(\mathcal{Q})$  is uniquely determined by its values on the generators,

$$\rho \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} 1 \quad 2 \end{array} & \begin{array}{c} \dots \end{array} & \begin{array}{c} m \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} 1 \quad 2 \end{array} & \begin{array}{c} \dots \end{array} & \begin{array}{c} n \end{array} \end{array} \right) = \sum \Gamma_{J_1, \dots, J_l}^{I_1, \dots, I_k} x^{J_1} \otimes \dots \otimes x^{J_l} \otimes e_{I_1} \otimes \dots \otimes e_{I_k}.$$

for some  $\Gamma_{J_1, \dots, J_l}^{I_1, \dots, I_k} \in \mathbb{K}$ . It is a straightforward calculation to check that the map  $i : \text{Rep}(\mathcal{Q}) \rightarrow C_{diff}^{\bullet, \bullet}(\bar{\mathcal{O}}_V)$  given by

$$i(\rho) := \sum \frac{\Gamma_{J_1, \dots, J_l}^{I_1, \dots, I_k}}{|I_1|! \cdots |J_l|!} x^{J_1} \otimes \dots \otimes x^{J_l} \cdot \Delta^{l-1} \left( \frac{\partial^{|I_1|}}{\partial x^{I_1}} \right) \cdots \Delta^{l-1} \left( \frac{\partial^{|I_l|}}{\partial x^{I_l}} \right)$$

satisfy the condition  $\rho \circ d = d_{GS} \circ \rho$ . Now 3.5.3(ii) follows immediately from isomorphism (9) and Proposition 3.5.1.  $\square$

**3.6 Gerstenhaber-Schack complex.** It is not hard to guess which dg prop,  $(\hat{\mathcal{Q}}, \hat{d})$ , has the property that its associated dg space of representations,  $(\text{Rep}(\hat{\mathcal{Q}}), \hat{d})$ , is exactly the Gerstenhaber-Schack complex  $(C^{\bullet, \bullet}(\bar{\mathcal{O}}_V), d_{GS})$ . As a prop,  $\hat{\mathcal{Q}}$  is, by definition, the same as  $\mathcal{Q}$  above, but the differential differs from  $d$  by the following four groups of terms,

$$(\hat{d} - d) \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} I_1 \end{array} & \begin{array}{c} I_k \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} J_1 \end{array} & \begin{array}{c} J_l \end{array} \end{array} \right) = - \sum_{\substack{J_{\bullet} = J'_1 \sqcup J''_1 \\ \sum |J'_i| = |I_1|}} \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} I_1 \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} \vdots \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} J'_1 \end{array} \end{array} \right) \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} I_2 \end{array} & \begin{array}{c} I_k \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} J''_1 \end{array} & \begin{array}{c} J''_l \end{array} \end{array} \right) - \sum_{\substack{J_{\bullet} = J'_1 \sqcup J''_1 \\ \sum |J''_i| = |I_k|}} (-1)^k \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} I_1 \end{array} & \begin{array}{c} I_{k-1} \end{array} & \begin{array}{c} I_k \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} & \begin{array}{c} \vdots \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} J'_1 \end{array} & \begin{array}{c} J'_l \end{array} \end{array} \right) \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} I_k \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} \vdots \end{array} \\ \vdots \quad \vdots \quad \vdots \\ \begin{array}{c} J''_1 \end{array} \end{array} \right)$$

$$\begin{aligned}
& - \sum_{\substack{I_\bullet = I'_\bullet \sqcup I''_\bullet \\ \sum |I'_\bullet| = |J_1|}} \left( \text{Diagram 1} \right) - \sum_{\substack{I_\bullet = I'_\bullet \sqcup I''_\bullet \\ \sum |I'_\bullet| = |J_l|}} (-1)^l \left( \text{Diagram 2} \right)
\end{aligned}$$

Using a spectral sequence argument very similar to the one used in the proof of Proposition 3.4.1, one easily obtains the following

**3.6.1 Proposition.**  $H(\hat{\mathcal{Q}}) = H(\mathcal{Q})$ .

**3.6.2 Corollary.**  $H(C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)) = H(C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)) = \wedge^{\bullet \geq 1} V \otimes \odot^{\bullet \geq 1} V^*$ .

The latter result together with the standard results from the theory of simplicial modules [Lo] imply formula (3).

**3.7 On the Etingof-Kazhdan quantization.** Note that the Gerstenhaber-Schack complex  $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$  has a structure of prop, the endomorphism prop of  $\bar{\mathcal{O}}_V$ . Moreover, it is easy to see that  $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$  is also closed under prop compositions so that the natural inclusion,  $j : C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V) \rightarrow C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$ , is a morphism of props [Me4]. A choice of a minimal resolution,  $\mathcal{AssB}_\infty$ , of the prop,  $\mathcal{AssB}$ , of associative bialgebras, induces [MV] on  $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$  (resp. on  $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ ) the structure of a filtered  $L_\infty$ -algebra whose Maurer-Cartan elements describe deformations of the standard bialgebra structure on  $\bar{\mathcal{O}}_V$  in the class of (resp. polydifferential) strongly homotopy bialgebra structures. Moreover [MV], the initial term of this induced  $L_\infty$ -structure is precisely the Gerstenhaber-Schack differential. The inclusion map  $j : C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V) \rightarrow C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$  extends to a morphism of  $L_\infty$ -algebras which, by isomorphisms (4) and (3), is a quasi-isomorphism. The Etingof-Kazhdan universal quantization [EK] of (possibly, infinite-dimensional) Lie bialgebra structures on  $V$  associates to such a structure, say  $\nu$ , a Maurer-Cartan element,  $\gamma_\nu$ , in the  $L_\infty$ -algebra  $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$ . As  $L_\infty$  quasi-isomorphisms are invertible [Ko], there is always an associated Maurer-Cartan element  $j^{-1}(\gamma_\nu^{EK})$  which, for degree reasons, describes an associated to  $\nu$  polydifferential bialgebra structure on  $\bar{\mathcal{O}}_V$ . Thus we proved that *for any Lie bialgebra structure,  $\nu$ , on a vector space  $V$  there exists its bialgebra quantization,  $j^{-1}(\gamma_\nu^{EK})$ , within the class of polydifferential operators from  $C^{\bullet,\bullet}_{poly}(\mathcal{O}_V)$ .*

## 4 Dg prop of unital $A_\infty$ -structures

**4.1 Differential in a free prop.** A differential in a free prop  $\mathcal{F}^\dagger\langle E \rangle$  can be decomposed into a sum,  $\delta = \sum_{p \geq 1} \delta_{(p)}$ , where  $\delta^{(p)} : E \xrightarrow{\delta} \mathcal{F}^\dagger\langle E \rangle \xrightarrow{pr_p} \mathcal{F}^\dagger_{(p)}\langle E \rangle$  is the composition of  $\delta$  with the projection to the subspace,  $\mathcal{F}^\dagger_{(p)}\langle E \rangle \subset \mathcal{F}^\dagger\langle E \rangle$ , spanned by decorated graphs with precisely  $p$  vertices. We studied in §3 free props equipped with differentials of the form  $\delta = \delta_{(1)}$  which preserve the number of vertices of decorated graphs, and heavily used the fact that  $\delta$  makes the associated space of representations,  $\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle) \simeq \text{Hom}(E, \mathcal{E}nd_V)$ , into a complex whose cohomology one can easily read from the cohomology of

$(\mathcal{F}^\dagger\langle E \rangle, \delta)$ . Remarkably [MV], a generic differential  $\delta$  in  $\mathcal{F}^\dagger\langle E \rangle$  makes the vector space  $\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle)[1]$  into a  $L_\infty$ -algebra whose  $p$ th homotopy Lie bracket is completely determined by  $p$ -th summand,  $\delta_{(p)}$ , of the differential  $\delta$ . In particular, if  $\delta$  has  $\delta_{(p)} = 0$  for all  $p \geq 3$ , then  $\text{Rep}_V(\mathcal{F}^\dagger\langle E \rangle)[1]$  is canonically a dg Lie algebra with the differential determined by  $\delta_{(1)}$  and Lie brackets determined by  $\delta_{(2)}$ . Thus, if we want to extend isomorphisms (1) and (2) into isomorphisms of Lie algebras, we have to look for more complicated (than the ones studied in §3) dg props canonically associated with the (polydifferential) Hochschild complex for  $\mathcal{O}_V$ .

**4.2 Dg prop of polyvector fields.** Let  $\mathcal{Poly}\mathcal{V}$  be a dg free prop generated by the  $\mathbb{S}$ -module,  $X[-1] = \{X(m, n)[-1]\}_{m \geq 1, n \geq 0}$ ,

$$X(m, n)[-1] = \text{sgn}_m \otimes \mathbb{1}_n[m-2] = \text{span} \langle \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 2 \quad \dots \quad n \end{array} \rangle$$

which is obtained from the  $\mathbb{S}$ -module  $X$  of Proposition 3.3.1 by a degree shift. The differential in  $\mathcal{Poly}\mathcal{V}$  is defined as follows (cf. [Me1]),

$$\partial \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 2 \quad \dots \quad n \end{array} = \sum_{\substack{[m] = I_1 \sqcup I_2 \\ [n] = J_1 \sqcup J_2 \\ |I_1| \geq 0, |I_2| \geq 1 \\ |J_1| \geq 1, |J_2| \geq 0}} (-1)^{\sigma(I_1 \sqcup I_2) + |I_1|(|I_2| + 1)} \begin{array}{c} \overbrace{\begin{array}{c} \dots \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \dots \end{array}}^{I_1} \quad \overbrace{\begin{array}{c} \dots \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \dots \end{array}}^{I_2} \\ \underbrace{\hspace{10em}}_{J_1} \quad \underbrace{\hspace{10em}}_{J_2} \end{array}$$

where  $\sigma(I_1 \sqcup I_2)$  is the sign of the permutation  $[n] \rightarrow I_1 \sqcup I_2$ . This differential is quadratic,  $\partial = \partial_{(2)}$ , so that, according to the general theory (see Theorem 60 in [MV]), the space  $\text{Rep}(\mathcal{Poly}\mathcal{V})_V[1] = \wedge^{\bullet \geq 1} V \otimes \odot^\bullet V \simeq \wedge^{\bullet \geq 1} \mathcal{T}_V$  comes equipped with a Lie algebra structure which, as it is not hard to check (cf. [Me1]), is precisely the Schouten bracket.

**4.3 Unital  $A_\infty$ -structures on  $\mathcal{O}_V$ .** It is well-known that the vector space  $\bar{\mathcal{C}}^\bullet(\mathcal{O}_V) := \bigoplus_{k \geq 1} \text{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k]$  has a natural graded Lie algebra structure with respect to the Gerstenhaber brackets,  $[\cdot, \cdot]_G$ . By definition, an  $A_\infty$ -algebra structure on the space  $\mathcal{O}_V$  is a Maurer-Cartan element in this Lie algebra, that is, a total degree 1 element  $\Gamma \in \bar{\mathcal{C}}^\bullet(\mathcal{O}_V)$  such that  $[\Gamma, \Gamma]_G = 0$ . Such an element,  $\Gamma$ , is equivalent to a sequence of homogeneous linear maps,  $\{\Gamma_k : \mathcal{O}_V^{\otimes k} \rightarrow \mathcal{O}_V[2-k]\}$  satisfying a sequence of quadratic equations (cf. [St]). An  $A_\infty$ -algebra structure is called *unital* if, for every  $k \geq 3$ , the map  $\Gamma_k$  factors through the composition  $\mathcal{O}_V^{\otimes k} \rightarrow \bar{\mathcal{O}}_V^{\otimes k} \rightarrow \mathcal{O}_V[k-2]$  and  $\Gamma_2(1, f) = \Gamma_2(f, 1) = f$ . The following lemma is obvious.

**4.3.1 Lemma.** *There is a one-to-one correspondence between unital  $A_\infty$ -structures on  $\mathcal{O}_V$  and Maurer-Cartan elements,*

$$\{\Gamma \in \bar{\mathcal{C}}^\bullet(\bar{\mathcal{O}}_V) : |\Gamma| = 1 \text{ and } d_H \Gamma + \frac{1}{2}[\Gamma, \Gamma]_G = 0\},$$

*in the Hochschild dg Lie algebra,  $\bar{\mathcal{C}}^\bullet(\bar{\mathcal{O}}_V)$ , for the ring  $\bar{\mathcal{O}}_V \subset \mathcal{O}_V$ .*

It was shown in [Me2] that there is a one-to-one correspondence between degree 0 representations of the dg prop  $(\mathcal{D}ef \mathcal{Q}, d)$  in a dg vector space  $V$  and Maurer-Cartan elements in the Hochschild dg Lie algebra  $(\bar{C}^\bullet(\bar{\mathcal{O}}_V), [\_, \_]_G, d_H)$ , i.e. with unital  $A_\infty$ -structures on  $\mathcal{O}_V$ . Put another way, the dg Lie algebra induced on  $\text{Rep}_V(\mathcal{D}ef \mathcal{Q})[1]$  from the above differential  $d$  is precisely the Hochschild dg Lie algebra.

Consider now a filtration,  $F_{-p} := \{\text{span}\langle G \rangle : \text{number of vertices in } G \geq p\}$ , of the complex  $(\mathcal{D}ef \mathcal{Q}, d)$ . It is clear that 0-th term,  $(E_0, \delta)$ , of the associated spectral sequence,  $\{E_r, d_r\}_{r \geq 1}$ , isomorphic (modulo an inessential shift of degree) to the prop  $(\mathcal{D}, \delta)$  introduced in §3.3 so that, by Proposition 3.3.1, we conclude that  $E_1 = H(E_0)$  is isomorphic as a free prop to  $\mathcal{Poly}\mathcal{V}$  whose shifted representation space,  $\text{Rep}_V(\mathcal{Poly}\mathcal{V})[1]$ , is  $H(\tilde{C}^\bullet(\tilde{O}_V)) = \wedge^{\geq 1} \mathcal{T}_V$ . The Lie algebra structure on  $H(\tilde{C}^\bullet(\tilde{O}_V))$  induced from the Gerstenhaber brackets on  $\tilde{C}^\bullet(\tilde{O}_V)$  is then given by the differential,  $d_1$ , induced on the next term of the spectral sequence,  $E_1 = \mathcal{Poly}\mathcal{V}$ , from the differential  $d$  in  $\mathcal{D}ef \mathcal{Q}$ . A direct inspection of formula (18) implies that  $d_1$  is precisely  $\partial$  which in turn implies by §4.2 that the induced Lie algebra structure on  $H(\tilde{C}^\bullet(\tilde{O}_V))$  is indeed given by Schouten brackets. It is worth noting in conclusion that  $L_\infty$ -morphisms (in the sense of Kontsevich [Ko]) between dg Lie algebras  $\tilde{C}^\bullet(\tilde{O}_V)$  and  $\wedge^{\geq 1} \mathcal{T}_V$  can be equivalently understood as morphisms of dg props,  $\mathcal{D}ef \mathcal{Q} \rightarrow \mathcal{Poly}\mathcal{V}^\circ$ , where  $\mathcal{Poly}\mathcal{V}^\circ$  is the wheeled completion of the prop of polyvector fields (by definition,  $\mathcal{Poly}\mathcal{V}^\circ$  is the smallest wheeled prop containing  $\mathcal{Poly}\mathcal{V}$  as a subspace). This point of view on quantizations was discussed in more detail in [Me2, Me3].

<sup>3</sup>We have to assume that  $\mathcal{Def} \mathcal{Q}$  is completed with respect to the genus filtration.

## References

- Ba. Baranovsky, V.: A universal enveloping for  $L_\infty$ -algebras.. Preprint arXiv:0706.1396 (2006).
- BM. Borisov, D., Manin, Yu.I.: Internal cohomomorphisms for operads, arXiv:math.CT/0609748 (2006).
- CFL. Cattaneo, A.S., Fiorenza, D., Longoni, R.: On the Hochschild-Kostant-Rosenberg map for graded manifolds, arXiv:math/0503380 (2006).
- Ch. Chapoton, F.: Opérades différentielles graduées sur les simplexes et les permutoèdres . Preprint arXiv:math/0102172 (2001).
- GS1. Gerstenhaber, M., Schack, S.D.: Bialgebra cohomology, deformations, and quantum groups. Proc. Nat. Acad. Sci. USA, **87**, 478-481 (1990).
- GS2. Gerstenhaber, M., Schack, S.D.: Algebras, bialgebras, quantum groups, and algebraic deformations. In: Contemporary Mathematics **134**, 51-92 (1992).
- GK. Getzler E., Kapranov, M.M.: Modular operads. Compositio Math., **110**, 65-126 (1998).
- EK. Etingof, P., Kazhdan, D.: Quantization of Lie bialgebras, I. Selecta Math. (N.S.) **2**, 1-41 (1996)
- Ko. Kontsevich, M.: Deformation quantization of Poisson manifolds I, math/9709040.
- LM. Lazarev, A.Yu., Movshev, M.V.: Deformations of Hopf algebras, Uspekhi Mat. Nauk, **46**, 211-212 (1991)
- Lo. Loday, J.L.: Cyclic homology, Springer (1998)
- MSS. Markl, M., Shnider, S., Stasheff, J.D.: Operads in Algebra, Topology and Physics. American Mathematical Society, Providence, Rhode Island (2002)
- Me1. Merkulov, S.A.: Prop profile of Poisson geometry, math.DG/0401034, Commun. Math. Phys. **262**, 117-135 (2006)
- Me2. Merkulov, S.A.: Graph complexes with loops and wheels. To appear in: Algebra, Arithmetic and Geometry - Manin Festschrift (2007)
- Me3. Merkulov, S.A.: Lectures on props, Poisson geometry and deformation quantization, to appear in Poisson Geometry in Mathematics and Physics, Contemporary Mathematics (eds. G. Dito, J.H. Lu, Y. Maeda and A. Weinstein), AMS (2007).
- Me4. Merkulov, S.A.: Quantization of strongly homotopy Lie bialgebras, arXiv:math/0612431 (2006)
- MV. Merkulov, S.A., Vallette, B.: Deformation theory of representations of prop(erad)s, preprint arXiv:0707.0889 (2007).
- St. Stasheff, J.D.: On the homotopy associativity of  $H$ -spaces, I II, Trans. Amer. Math. Soc. **108**, 272-292 & 293-312 (1963)
- SU. Saneblidze, S., Umble, R.: Diagonals on the permutahedra, multiplihedra and associahedra. Homology Homotopy Appl. **6**, 363-411 (2004).
- Va. Vallette, B.: A Koszul duality for props. Trans. AMS., **359**, 4865-4943 (2007)